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Exceptional Families, Topological Degree and Complementarity Problems

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Abstract. By using the topological degree we introduce the concept of "exceptional family of elements" specifically for continuous functions. This has important consequences pertaining to the solvability of the explicit, the implicit and the general order complementarity problems. In this way a new direction for research in the complementarity theory is now opened.

Key words: Exceptional family of elements, topological degree and complementarity problems

1. Introduction

Complementarity theory is a new and interesting domain of applied mathematics. It is a link between such concepts as fixed point theory topological degree, variational inequalities, linear and nonlinear analysis and domains of applied mathematics like optimization, game theory, economics, classical mechanics and stochastic optimal control etc. [1], [10], [19].

It is associated with the idea of "equilibrium" as studied in physics, engineering and even economics [6–13]. Its implication to the study of global optimization is also important [21].

The relations between the complementarity theory and the global optimization is also an important aspect.

Several kinds of complementarity problems have been defined and are being studied. There are explicit and implicit complementarity problems, each of which can be considered with respect to a dual system or with respect to an ordering [6], [8], [9], [11], [12].

In this paper we introduce the concept of *exceptional family of elements* for a function with respect to a convex cone, and we establish some relations between this notion, the topological degree and the complementarity problem. The main results are several alternative theorems with applications to the solvability of the most important kind of general nonlinear complementarity problems.

It is remarkable that our existence results for complementarity problems are based only on the concepts of exceptional family of elements and continuity.

The fundamental idea of this paper is the relation between the concept of exceptional family of elements, the topological degree and the complementarity problem. The topological degree was also used by other authors in the study of complementarity problems [3–5]. In this paper we consider nonlinear complementarity problems only.

2. Preliminaries

The preliminaries are presented in a general Banach or Hilbert space, since the same notions can be used to generalize our results to an infinite dimensional Hilbert space.

Let $(E, \| \ \|)$ be a Banach space and let E^* be the topological dual of E. We denote by $\langle E, E^* \rangle$ a duality between E and E^*. We say that $\mathbf{K} \subset E$ is a closed convex cone if and only if $\mathbf{K}$ is a closed subset and the following properties are satisfied:

(1). $\mathbf{K} + \mathbf{K} \subseteq \mathbf{K}$

(2). $\lambda \mathbf{K} \subseteq \mathbf{K}$ for all $\lambda \in R_+$,

(3). $\mathbf{K} \cap (-\mathbf{K}) = \{0\}$.

Whenever a closed convex cone $\mathbf{K} \subset E$ is defined, we have an ordering on E defined by $x \leq y$, if and only if $y - x \in \mathbf{K}$. By definition the *dual* of $\mathbf{K}$ is $\mathbf{K}^* = \{y \in E^* | \langle x, y \rangle \geq 0$ *for all* $x \in \mathbf{K}\}$. Note that $\mathbf{K}^*$ is also a closed convex cone. We say that the ordered Banacgh space $(E, \| \ \|, \mathbf{K})$ is a *vector lattice*, if and only if, for every pair (x, y) of elements of E, the supremum $x \vee y$ and the infimum $x \wedge y$ exist in E. If $(E, \| \ \|, \mathbf{K})$ is a vector lattice we define for every $x \in E, x^+ = x \vee 0, x^- = (-x) \vee 0$, and $|x| = x^+ + x^-$. Other properties of x^+, x^- and $|x|$ are presented and proved in [22]. Suppose now that $(E, \| \ \|)$ is a Hilbert space denoted by $(H, \langle , \rangle)$ where $\langle , \rangle$ is the inner product defined on H.

We say that an ordered Hilbert space $(H, \langle , \rangle, \mathbf{K})$ is a *Hilbert lattice* if and only if:

(h_1) H is a vector lattice,

(h_2) $\| |x| \| = \|x\|$ for every $x \in H$

(h_3) $0 \leq x \leq y$ implies $\|x\| \leq \|y\|$ for every $x, y \in \mathbf{K}$.

If $D \subset H$ is a closed convex set we denote the projection onto D by P_D that is, for every $x \in H, P_D(x)$ is the unique element satisfying,

$$\|x - P_D(x)\| = \min_{y \in D} \|x - y\|.$$

If $\mathbf{K} \subset H$ is a closed convex cone we denote the projection onto $\mathbf{K}$ by $P_{\mathbf{K}}$.

The *polar cone* of $\mathbf{K}$ is by definition $\mathbf{K}^0 = \{x \in H | \langle x, y \rangle \leq 0$ for all $y \in \mathbf{K}\}$; polarity is studied in the book [22]. If $\mathbf{K}$ and $\mathbf{Q}$ are two closed convex cones in

H then **K** and **Q** are *mutually polar* if $\mathbf{K} = \mathbf{Q}^0$. We recall the following classical result.

THEOREM 1 ([Moreau] [18]). *If* **K** *and* **Q** *are two mutually polar convex cones in a Hilbert space* **H** *and* $x, y, z \in \mathbf{H}$*, then the following statements are equivalent:*

$$(i) \quad z = x + y, x \in \mathbf{K}, y \in \mathbf{Q} \text{ and } \langle x, y \rangle = 0,$$
$$(ii) \quad x = P_{\mathbf{K}}(z) \text{ and } y = P_Q(z).$$

The operator $P_{\mathbf{K}}$ has several special properties and the following notion was defined and studied in [13].

We say that K is an *isotone projection cone* if and only if, for every $x, y \in H, x \leq y$ implies that $P_{\mathbf{K}}(x) \leq P_{\mathbf{K}}(y)$. The following result is proved in [13].

THEOREM 2. *If* $(\mathbf{H}, \langle, \rangle, \mathbf{K})$ *is a Hilbert lattice then* **K** *is an isotone projection cone and moreover,* $P_{\mathbf{K}}(x) = x^+$ *for every* $x \in \mathbf{H}$.

This result justifies some of our notations and also, by this result we have that for the Euclidean space $(\mathbf{R}^n, \langle, \rangle), P_{\mathbf{R}^n_+}(x) = x^+$ for every $x \in \mathbf{R}^n$.

The principal aim of our paper is the study of the following three kinds of complementarity problems.

Let $\langle E, E^* \rangle$ be a duality of Banach spaces and let $\mathbf{K} \subset E$ be a closed convex cone. Given the mappings $f : \mathbf{K} \to E^*; g : D \to E$, where $D \subset E$ is a subset, we consider the following complementarity problems:

$$ECP(f, \mathbf{K}) : \begin{cases} \text{find } x_0 \in \mathbf{K} \text{ such that,} \\ f(x_0) \in \mathbf{K}^* \text{ and } \langle x_0, f(x_0) \rangle = 0 \end{cases}$$

$$ICP(f, g, D, \mathbf{K}) : \begin{cases} \text{find } x_0 \in D \text{ such that,} \\ g(x_0) \in \mathbf{K}, f(x_0) \in \mathbf{K}^* \text{ and} \\ \langle g(x_0), f(x_0) \rangle = 0 \end{cases}$$

When $D = \mathbf{K}$ we denote the last problem by $ICP(f, g, \mathbf{K})$. If E is a vector lattice with respect to the ordering defined by **K** and $f_1, f_2, \dots f_n$ are mappings from E into E we consider the problem

$$GOCP(\{f_i\}_{i=1}^n, \mathbf{K}) : \begin{cases} \text{find } x_0 \in \mathbf{K} \text{ such that} \\ \wedge(f_1(x_0), f_2(x_0), \dots f_n(x_0)) = 0 \end{cases}$$

We say that $ECP(f, \mathbf{K})$ is the *explicit complementarity problem*, $ICP(f, g, \mathbf{K})$ is the *Implicit complementarity problem* and $GOCP(\{f_i\}_{i=1}^n, \mathbf{K})$ the *generalized order complementarity problem*. Suppose that $(\mathbf{H}, \langle, \rangle)$ is a Hilbert space and $\mathbf{K} \subset \mathbf{H}$ a closed convex cone. The following result is proved in [7].

PROPOSITION 2 ([7]). *Given the mapping* $f : \mathbf{K} \to H$, *the complementarity problem* $ECP(f, \mathbf{K})$ *has a solution, if and only if, the mapping*

$$\Psi(x) = P_{\mathbf{K}}(x) - f(P_{\mathbf{K}}(x));\ \textit{for all } x \in \mathbf{H} \tag{1}$$

has a fixed point in $\mathbf{H}$. *If* x_0 *is a fixed point of* Ψ *then* $x_* = P_{\mathbf{K}}(x_0)$ *is a solution of the problem* $ECP(f, \mathbf{K})$.

Proof. The proof in [7] is based on Moreau's Theorem and on the fact that the projection operator $P_{\mathbf{K}}$ is characterized by the following result. For every $x \in H, P_{\mathbf{K}}(x)$ is the element of $\mathbf{K}$ satisfying the following conditions:

$$(i) \quad \langle P_{\mathbf{K}}(x) - x, y\rangle \geq 0,\ \textit{for all } y \in \mathbf{K}$$
$$(ii) \quad \langle P_{\mathbf{K}}(x) - x, P_{\mathbf{K}}(x)\rangle = 0. \qquad \blacksquare$$

In this paper we use systematically the topological degree as it is presented in the books [15], [16], [23].

Without other specifications, we will denote by D a bounded open subset of $\mathbf{R}^n$ and by y an arbitrary point of $\mathbf{R}^n$. The closure of D is written $\bar{D}$ and its boundary ∂D. We denote by $\mathcal{C}(\bar{D})$ the linear space of continuous functions from $\bar{D}$ into $\mathbf{R}^n$. If $F \in \mathcal{C}(\bar{D})$ and $y \in \mathbf{R}^n$ such that $y \notin F(\partial D)$ we denote by $\deg(F, D, y)$ the degree associated with F, D and y.

If $F, G \in \mathcal{C}(\bar{D})$ we consider the homotopy $H(x,t) = tG(x) + (1-t)F(x), 0 \leq t \leq 1$.

THEOREM 3 ([Poincaré–Bohl][16][23]). *Let* $D \subset \mathbf{R}^n$ *be an open bounded subset and* $F, G \in \mathcal{C}(\bar{D})$ *two continuous mappings. If* $y \in \mathbf{R}^n$ *is an arbitrary point satisfying the condition*

$$y \notin \{H(x,t) | x \in \partial D \textit{ and } t \in [0,1]\} \tag{2}$$

then deg$(G, D, y) =$ *deg*(F, D, y). $\blacksquare$

3. Exceptional Families

We consider in this section the space $\mathbf{R}^n$ endowed with the Euclidean structure. First, consider the closed convex cone $\mathbf{K} = \mathbf{R}^n_+ = \{x = (x_i) \in \mathbf{R}^n | x_i \geq 0, i = 1, 2, \ldots, n\}$. Let $f : \mathbf{R}^n_+ \to \mathbf{R}^n$ be a continuous mapping.

DEFINITION 1. A set of points $\{x^r\}_{r>0} \subset \mathbf{R}^n_+$ is an exceptional family of elements for f if $\|x^t\| \to +\infty$ as $r \to +\infty$ and for each $r > 0$ there exists $\mu_r > 0$ such that

$$(i) \quad f_i(x^r) = -\mu_r x_i^r \quad \text{if} \quad x_i^r > 0$$
$$(ii) \quad f_i(x^r) \geq 0 \quad \text{if} \quad x_i^r = 0$$

DEFINITION 2. An exceptional family of elements for f is regular if $\|x^r\| = r$ for every $r > 0$.

Now consider the general case. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a continuous mapping and $\mathbf{K} \subset \mathbf{R}^n$ a *closed convex cone*. If $\mathbf{Q} = \mathbf{K}^0$, then by the bipolarity Theorem [22] it follows that $\mathbf{K} = \bar{\mathbf{K}} = \mathbf{Q}^0$ and hence $\mathbf{K}$ and $\mathbf{Q}$ are mutually polar. By Moreau's Theorem each vector $z \in \mathbf{R}^n$ has a unique representation of the form

$$z = z^+ - z^- \tag{3}$$

where $z^+ = P_{\mathbf{K}}(z)$ and $z^- = -P_{\mathbf{K}^0}(z)$. (Note that $-z^-$ is the orthogonal complement to z^+). Also, it follows that $z^- = z^+ - z$.

DEFINITION 3. A set of points $\{x^r\}_{r>0} \subset \mathbf{R}^n$ is an exceptional family of elements for f, (with respect to $\mathbf{K}$) if $\|(x^r)^+\| \to +\infty$ as $r \to +\infty$, and for each $r > 0$ the point $f((x^r)^+)$ belongs to open ray

$$\mathcal{O}((x^r)^-; s_r) = \{y = (x^r)^- + \mu s_r | \mu > 0\}; \tag{4}$$

where $s_r = (x^r)^- - (x^r)^+$.

REMARK. If, in particular, $x^r \in \mathbf{K}$, then from (4) we have the equality

$$f(x^r) = -\mu_r(x^r) \tag{5}$$

for some $\mu_r > 0$. Considering $\mathbf{K} = \mathbf{R}^n_+$ in *Definition 3* we do not obtain exactly *Definition 1*. In *Definition 1* there is more information about $f_i(x^r)$ because of the particularities of the cone $\mathbf{R}^n_+$. For a general cone $\mathbf{K} \subset \mathbf{R}^n$ there is also the concept of *regular exceptional family of elements*.

DEFINITION 4. An exceptional family of elements for f (with respect to $\mathbf{K}$) is regular if $\|(x^r)^+\| = r$, for every $r > 0$.

The concept of *regular exceptional family of elements* was independently discovered by T.E. Smith [24] under the name of *exceptional sequence* and a few years ago, by G. Isac [unpublished notes] under the name of *opposite radial sequence*. The concept of *exceptional family of elements* introduced in this paper is a new concept, and it is less restrictive than the concept of regular exceptional family of elements. We define now the concept of exceptional family of elements for a couple of mappings. Let $f, g : \mathbf{R}^n \to \mathbf{R}^n$ be continuous mappings.

DEFINITION 5. A set of points $\{x_r\}_{r>0} \subset \mathbf{R}^n$ is an exceptional family of elements for the couple (f, g) if $\|x^r\| \to +\infty$ as $r \to +\infty$, $g(x^r) \geq 0$ for each $r > 0$ and there exists $\mu_r > 0$ such that for $i = 1, 2, \ldots, n$

$$(a) \quad f_i(x^r) = -\mu_r(g_i)(x^r), \quad \text{if} \quad g_i(x^r) > 0;$$
$$(b) \quad f_i(x^r) \geq 0, \quad \text{if} \quad g_i(x^r) = 0.$$

REMARK. As the problem $ICP(f, g, \mathbf{R}^n_+)$ is symmetrical with respect to the mappings f and g, we can give an analogous definition of an exceptional family of elements for the pair (g, f).

The concept of *exceptional family of elements* is close to the complementarity problem; for example, one of the main results is the following. *If* $f : \mathbf{R}^n_+ \to \mathbf{R}^n$ *is a continuous function then there exists either a solution for the problem* $CP(f, \mathbf{R}^n_+)$ *or an exceptional family of elements for* f. Hence, every continuous mapping $f : \mathbf{R}^n_+ \to \mathbf{R}^n$ with the property that the problem $CP(f, \mathbf{R}^n_+)$ has no solutions, has an exceptional family of elements. Since, our aim is to obtain existence theorems for complementarity problems, it is interesting to know *when a continuous function does not have an exceptional family of elements*.

We have some interesting results for *regular exceptional family of elements*. The followint classical results about rthe projection operator onto a closed convex set in a Hilbert space are useful [14]. Let D be a closed convex set in a Hilbert space $(H, \langle , \rangle)$. If $x \in \mathbf{H}$, then $y = P_D(x)$ if and only if

$$\begin{cases} y \in D \textit{ and} \\ \langle y, v - y\rangle \geq \langle x, v - y\rangle \textit{ for all } v \in D; \end{cases} \tag{6}$$

Let $f : D \to H$ be a mapping. The problem

$$\begin{cases} \textit{find } x \in D \textit{ such that} \\ \langle f(x), y - x\rangle \geq \ \textit{for all } y \in D, \end{cases} \tag{7}$$

is equivalent to the problem

$$\begin{cases} \textit{find } x \in D \textit{ such that} \\ \langle x, y - x\rangle \geq \langle x - f(x), y - x\rangle \textit{ for all } y \in D, \end{cases} \tag{8}$$

and by (6) we have that problem (7) is equivalent to the fixed point problem

$$\begin{cases} \textit{find } x \in D \textit{ such that} \\ x = P_D(x - f(x)), \end{cases} \tag{9}$$

If $(H, \langle , \rangle)$ is the Euclidean space $(\mathbf{R}^n, \langle , \rangle)$ for every $r > 0$ we denote $[\mathbf{R}^n_+]_r = \{x \in \mathbf{R}^n_+ | \|x\| \leq r\}$ and the projection onto $[\mathbf{R}^n_+]_r$ of an arbitrary element $z \in \mathbf{R}^n$ will be denoted by $P_r(z)$. By the definition of $P_r(z)$ it follows that $P_r(z)$ is the solution of the differentiable convex program.

$$\begin{cases} \textit{minimize } \langle z - x, z - x\rangle \\ \textit{over the set } \{x \in \mathbf{R}^n_+ | \langle x, x\rangle \leq r^2\}. \end{cases}$$

Applying the Karush–Kuhn–Tucker optimality conditions to the Lagrangian $L(x, \mu) = \langle z - x, z - x\rangle + \mu(\langle x, x\rangle - r^2)$ the following result is obtained. *For* *each* $r > 0$ *and* $z \in \mathbf{R}^n$ *we have that* $x \in [\mathbf{R}^n_+]_r$ *is such that* $x = P_r(z)$ *if and only if, there exists* $\mu \geq 0$ *such that the following conditions are satisfied:*

$$(p_1) \quad x_i = 0 \Rightarrow (1+\mu)x_i \geq z_i$$
$$(p_2) \quad x_i > 0 \Rightarrow (1+\mu)x_i = z_i$$
$$(p_3) \quad \mu > 0 \Rightarrow \|x\| = r$$

If $f : \mathbf{R}^n_+ \to \mathbf{R}$ is a continuous function and $\{x^r\}_{r>0}$ is a *regular exceptional family of elements for* f then we can show that for every $r > 0$, the element x^r verifies $(p_1), (p_2), (p_3)$ with $z = x^r - f(x^r)$.

Thus, for every regular exceptional family $\{x^r\}_{r>0}$ for f we have that:

$$(i) \quad x^r = P_r(x^r - f(x^r)),$$
$$(ii) \quad \|x^r\| = r.$$

Using (9) and (7) it can be shown that for every regular exceptional family of elements of f the following conditions are satisfied:

$$(iii) \quad \langle f(x^r), y - x^r \rangle \geq 0 \text{ for all } y \in [\mathbf{R}^n_+]_r$$
$$(iv) \quad \|x^r\| = r.$$

Recall that $f : \mathbf{R}^n_+ \to \mathbf{R}^n$ is *coercive* on $\mathbf{R}^n_+$ if and only if

$$\frac{\langle f(x) - f(x_0), x - x_0 \rangle}{\|x - x_0\|} \to +\infty \textit{ as} \|x\| \to +\infty, x \in \mathbf{R}^n_+, \textit{ for some } x_0 \in \mathbf{R}^n_+.$$

The next result is to show that the class of functions without exceptional family of elements is nonempty.

PROPOSITION 4. *Every coercive continuous function* $f : \mathbf{R}^n_+ \to \mathbf{R}^n$ *does not have regular exceptional families of elements.*

Proof. Let $\{x^r\}_{r>0}$ be a regular exceptional family for f. For every r the properties iii) and iv) are satisfied. Since f is coercive we can take $\rho > \|f(x_0)\|$ *and* $r_0 > \|x_0\|$ such that $\langle f(x) - f(x_0), x - x_0 \rangle \geq \rho\|x - x_0\|$, for every x such that $\|x\| > r_0, x \in \mathbf{R}^n_+$. Thus, for every $r > r_0$ we have $\langle f(x) - f(x_0), x - x_0 \rangle \geq \rho\|x - x_0\|$ for every $x \in \mathbf{R}^n_+$ such that $\|x\| \geq r$. This implies that

$$\begin{cases} \langle f(x), x - x_0 \rangle \geq \rho\|x - x_0\| + \langle f(x_0), x - x_0 \rangle \geq \\ (\rho - \|f(x_0)\|)(\|x\| - \|x_0\|) > 0 \textit{ for every } x \in \mathbf{R}^n_+ \\ \text{with } \|x\| \geq r. \end{cases} \tag{10}$$

But, since x^r satisfies (iii) we have

$$\langle f(x^r), x^r - x_0 \rangle = -\langle f(x^r), x_0 - x^r \rangle \leq 0$$

and from (10) we must have $\|x^r\| \geq r$, that is $\|x^r\| < r$ which is a contradiction of (iv). Hence, it is impossible to have a regular exceptional family of elements for f. ■

The coercive functions were used in the study of complementarity problems in [17].

Considering the last result, we deduce that it is important to know if there exists a noncoercive continuous function without regular exceptional families of elements. Obviously, if $f : \mathbf{R}^n_+ \to \mathbf{R}^n$ is a continuous function such that $\|x - f(x)\| < \|x\|$ for all $x \in \mathbf{R}^n_+\backslash\{0\}$, then f is without regular exceptional families of elements. Indeed, this fact is a consequence of properties (i) and (ii), since for every $r > 0$ we have

$$\begin{aligned}\|P_r(x - f(x))\| &= \|P_r(x - f(x)) - P_r(0)\| \leq \|x - f(x)\| \\ &< \|x\| \textit{ for all } x \in \mathbf{R}^n_+\backslash\{0\}\end{aligned}$$

and every regular exceptional family of elements for f must satisfy (i) and (ii) which is impossible. We consider now an example of a continuous function which is noncoercive and without regular exceptional familes of elements. Let $f : \mathbf{R}^n_+ \to \mathbb{R}^h$ be the function defined by $f(x) = \frac{x}{\|x\|+1}$ for all $x \in \mathbf{R}^n_+$. The function f is continuous and for every $x \in \mathbf{R}^n_+\backslash\{0\}$ we have

$$\begin{aligned}\|x - f(x)\| &= \left\|x - \frac{x}{\|x\| + 1}\right\| = \left\|\left(1 - \frac{1}{\|x\| + 1}\right) x\right\| \\ &= \left(1 - \frac{1}{\|x\| + 1}\right) \|x\| < \|x\|.\end{aligned}$$

Thus, for every $x \in \mathbf{R}^n_+\backslash\{0\}$ we have $\|x - f(x)\| < \|x\|$ which implies that f does not have regular exceptional families of elements. We show now that f is not coercive.

Indeed, for $x_0 = 0$ we have

$$\begin{aligned}&\frac{\langle f(x) - 0, x - 0\rangle}{\|x\|} = \frac{\langle f(x), x\rangle}{\|x\|} = \frac{\|x\|^2}{\|x\|(\|x\| + 1)} \quad \textit{and} \\ &\frac{\langle f(x), x\rangle}{\|x\|} \to +\infty \textit{ as } \|x\| \to +\infty\end{aligned}$$

with $x \in \mathbf{R}^n_+$.

Suppose now that $x_0 \neq 0$ and consider the expression

$$\frac{\langle f(x) - f(x_0), x - x_0\rangle}{\|x - x_0\|}.$$

Then

$$\begin{aligned}\frac{\langle f(x) - f(x_0), x - x_0\rangle}{\|x - x_0\|} &= \frac{1}{\|x - x_0\|} \left\langle \frac{x}{\|x\| + 1} - \frac{x_0}{\|x_0\| + 1}, x - x_0 \right\rangle \\ &= \frac{1}{\|x - x_0\|} \left[\frac{\langle x, x\rangle}{\|x\| + 1} - \frac{\langle x, x_0\rangle}{\|x\| + 1} - \frac{\langle x, x_0\rangle}{\|x_0\| + 1} + \frac{\langle x_0, x_0\rangle}{\|x_0\| + 1}\right]\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\|x-x_0\|}\left[\frac{\|x\|^2}{\|x\|+1} - \left(\frac{1}{\|x\|+1}+\frac{1}{\|x_0\|+1}\right)\langle x, x_0\rangle + \frac{\|x_0\|^2}{\|x_0\|+1}\right] \\
&\geq \frac{1}{\|x\|+\|x_0\|}\left[\frac{\|x\|^2}{\|x\|+1} - \left(\frac{1}{\|x\|+1}+\frac{1}{\|x_0\|+1}\right)\|x\|\|x_0\| + \frac{\|x_0\|^2}{\|x_0\|+1}\right] \\
&= \frac{1}{\|x\|+\|x_0\|}\left[\frac{\|x\|^2}{\|x\|+1} - \frac{\|x\|\|x_0\|}{\|x\|+1} - \frac{\|x\|\|x_0\|}{\|x_0\|+1} + \frac{\|x_0\|^2}{\|x_0\|+1}\right] \\
&= \frac{\|x\|^2}{(\|x\|+\|x_0\|)(\|x\|+1)} - \frac{\|x\|\|x_0\|}{(\|x\|+\|x_0\|)(\|x\|+1)} \\
&\quad - \frac{\|x\|\|x_0\|}{(\|x_0\|+1)(\|x\|+\|x_0\|)} + \frac{\|x_0\|^2}{(\|x\|+\|x_0\|)(\|x_0\|+1)}.
\end{aligned}
$$

Computing the limit as $\|x\| \to +\infty$ we obtain

$$
\lim_{\|x\|\to+\infty} \frac{\langle f(x)-f(x_0), x-x_0\rangle}{\|x-x_0\|} = 1 - \frac{\|x_0\|}{\|x_0\|+1} \neq +\infty
$$

which means that f is noncoercive.

Another interesting example of a function without exceptional families of elements with respect to $\mathbf{R}^n_+$ is the following.

Consider an arbitrary continuous mapping $T : \mathbf{R}^n_+ \to \mathbf{R}^n$ such that $P_{\mathbf{K}}(T(x)) = 0$ for every $x \in \mathbf{R}^n_+$ and $S : \mathbf{R}^n_+ \to \mathbf{R}^n$ an arbitrary continuous mapping such that $\|S(x)\| < \|x\|$ for every $x \in \mathbf{R}^n_+$. The mapping $f(x) = x - S(x) + T(x)$ is without exceptional families of elements with respect to $\mathbf{R}^n_+$. Indeed, we have

$$
\begin{aligned}
\|P_r(x - f(x))\| &= \|P_r(x - f(x)) - P_r(T(x))\| \\
&\leq \|x - [x - S(x) + T(x) - T(x)]\| \leq \|S(x)\| < \|x\|
\end{aligned}
$$

for every $x \in \mathbf{R}^n_+$ and hence the conditions (i) and (ii) are not satisfied.

REMARK. The problem in which there exists a noncoercive function without regular exceptional families of elements, was considered in Proposition 4.7 of the paper [24], but the example presented in the proof of this proposition is incorrect, since a function F satisfying the both relations, $\|x - F(x)\| < \|x\|$ and $\langle F(x), x\rangle \leq 0$ is necessarily equal to zero for every $x \in \mathbf{R}^n_+$.

We conclude that the class of continuous functions without exceptional families of elements is strictly bigger than the class of coercive functions.

4. Applications to Complementarity Problems

Now, we apply these concepts to the study of complementarity problems. Our presentation follows the passage from the particular case to the general case. We consider first, the case of the Euclidean space $(\mathbf{R}^n, \langle , \rangle)$ ordered by the closed convex cone $\mathbf{R}^n_+$ which is self-adjoint.

Let $f : \mathbf{R}^n_+ \to \mathbf{R}^n$ be a continuous mapping and consider the nonlinear complementarity problem

$$ECP(f, \mathbf{R}^n_+) : \begin{cases} \textit{find } x_0 \in \mathbf{R}^n_+ \textit{ such that} \\ f(x_0) \in \mathbf{R}^n_+ \textit{ and } \langle x_0, f(x_0)\rangle = 0 \end{cases}$$

THEOREM 5. *For any continuous mapping $f : \mathbf{R}^n_+ \to \mathbf{R}^n$, there exists either a solution for the problem $ECP(f, \mathbf{R}^n_+)$ or an exceptional family of elements for f.*

Proof. Using *Proposition 2* we know that the solvability of the problem ECP $(f, \mathbf{R}^n_+)$ is equivalent to the problem of finding a fixed point for the mapping

$$\Psi(x) = P_{\mathbf{R}^n_+}(x) - f(P_{\mathbf{R}^n_+}(x)), (x \in \mathbf{R}^n).$$

Hence, we consider the equation $x = \Psi(x)$ or

$$f(P_{\mathbf{R}^n_+}(x)) + x - P_{\mathbf{R}^n_+}(x) = 0 \tag{11}$$

Since $(\mathbf{R}^n, \mathbf{R}^n_+)$ is a Hilbert lattice we know that $P_{\mathbf{R}^n_+}(x) = x^+$ (see *Theorem 2* and the paper [13]), and because $x - x^+ = -x^-$ the equation (11) becomes

$$f(x^+) - x^- = 0. \tag{12}$$

If we denote $F(x) = f(x^+) - x^-$ the problem is now to solve the equation

$$F(x) = f(x^+) - x^- = 0. \tag{13}$$

We examine problem (13) in detail. Obviously, the mapping $F : \mathbf{R}^n \to \mathbf{R}^n$ is continuous, Let $\{S_r\}_{r>0}$ be the family of spheres of radius r:

$$S_r = \{x \in \mathbf{R}^n | \|x\| = r\} \tag{14}$$

and B_r the open ball of radius r:

$$B_t = \{x \in \mathbf{R}^n | \|x\| < r\}. \tag{15}$$

We consider the homotopy between the identity mapping I and F, i.e.

$$H(x,t) = tx + (1-t)F(x); 0 \leq t \leq 1 \tag{16}$$

for an arbitrary $x \in \partial B_r = S_r$, and we apply *Theorem 3* [Poincaré–Bohl] with $y = 0$ and B_r for D. This gives

$$\begin{aligned} H(x,t) &= tx + (1-t)f(x^+) - (1-t)x^- \\ &= t(x + x^-) + (1-t)f(x^+) - x^- \\ &= tx^+ + (1-t)f(x^+) - x^-. \end{aligned} \tag{18}$$

Two cases are possible:
(i) There exists an $r > 0$ such that

$$0 \notin H(x,t), x \in S_r, t \in [0,1].$$

Then *Theorem 3* implies that

$$\deg(F, B_r, 0) = \deg(I, B_r, 0). \tag{19}$$

It is well known that $\deg(I, B_r, 0) = 1$, (cf. [16], [15], [20], [23](. Hence, $\deg(F, B_r, 0) = 1$ also.

This means that the ball $\bar{B}_r$ contains at least one solution to the equation $F(x) = 0$ [cf. Kronecker's Theorem (Theorem 2.1.1. of [16])]. Therefore the problem $ECP(f, \mathbf{R}^n_+)$ has a solution.
(ii) For each $r > 0$ there exist a point $u_r \in S_r$ and a scalar $t_t \in [0, 1)$ such that

$$H(u_r, t_r) = 0. \tag{20}$$

We remark that $\|u_r\|^2 = \langle u_r^+ - u_r^-, u_r^+ - u_r^- \rangle = \|u_r^+\|^2 + \|u_r^-\|^2 = r^2$. If $t_r = 0$, then u_r solves equation (13), which implies again that the problem $ECP(f, \mathbf{R}^n_+)$ has a solution. Otherwise, if $t_r > 0$, then (18) and (20) yield

$$t_r u_r^+ + (1 - t_r) f(u_r^+) = u_r^-. \tag{21}$$

From (21) we have

$$(1 - t_r) f_i(u_r^+) = -t_r (u_r^+)_i, \textit{if} (u_r)_i > 0 \tag{22}$$

and

$$(1 - t_r) f_i(u_r^+) = (u_r^-)_i, \textit{if} (u_r)_i \leq 0. \tag{23}$$

Now we put $x^r = u_r^+$ and we rearrange (22) and (23) as follows:

$$f_i(x^r) = -\frac{t_r}{1 - t_r} x_i^r, \ \textit{if}\, x_i^r > 0; \tag{24}$$

and

$$f_i(x^r) = \frac{1}{1 - t_r} (u_r^-)_i \geq 0, \ \textit{if}\, x_i^r = 0; \tag{25}$$

We take $\mu_r = \frac{t_r}{1 - t_r}$ and note that (24) and (25) represent relations (i) and (ii) from *Definition 1*. In order to demonstrate that $\{x^r\}_{r>0}$ is an exceptional family of elements, we must show that $\|x^r\| \to +\infty$ when $r \to +\infty$. On the contrary, we suppose the set $\{u_r^+\}$ to be bounded. In this case, it follows that

$$\|u_r^-\| = \sqrt{r^2 - \|u_r^+\|^2} \to +\infty,$$

which means that the right-hand sice of (21) is unbounded. On the other hand, the left-hand side of (21) is bounded since the set $\{u_r^+\}$ is supposed to be bounded and f is a continuous function. This contradiction completes the proof. ■

There is a similar results for regular exceptional families.

THEOREM 6. *For any continuous mapping* $f : \mathbf{R}^n_+ \to \mathbf{R}^n$ *there exists either a solution for the problem* $ECP(f, \mathbf{R}^n_+)$ *or a regular exceptional family of elements for* f.

Proof. Consider again the equation

$$F(x) = f(x^+) - x^- = 0. \tag{26}$$

For $r > 0$ define the set

$$D_r = W_r \cap \bar{B}_r \tag{27}$$

where

$$W_r = \{x \in \mathbf{R}^n | \|x^+\| \leq r\}, \tag{28}$$

and the number

$$\delta = \sqrt{(\max\{r, M_r\})^2 + r^2} + 1 \tag{29}$$

with

$$M_r = \max_{x \in W_r} \|f(x^+)\|. \tag{30}$$

It is easy to show that

$$M_r \leq \max_{x \in \bar{B}_r \cap \mathbf{R}^n_+} \|f(x)\| < +\infty$$

and hence we have that δ is well defined. As in the proof of *Theorem 5* we apply again *Theorem 3* to the mappings I, F and to the set D_r. It is sufficient to consider two cases:

(i) There exists an $r > 0$ such that

$$0 \notin H(x, t), x \in \partial D_r, t \in [0, 1).$$

Then, repeating the arguments used in the proof of *Theorem 5*, we obtain a solution of the problem $ECP(f, \mathbf{R}^n_+)$.

(ii) For each $r > 0$ there exist a point $u_r \in \partial D_r$ and a scalar $t_r \in [0, 1)$ such that

$$H(u_r, t_r) = 0. \tag{31}$$

If $t_r = 0$, then u_r is a solution of equation (26) which implies that the problem $ECP(f, \mathbf{R}^n_+)$ has a solution.

Otherwise, if $t_r > 0$, we obtain from the relations (21)–(25) that $x^r = u_r^+$ satisfies conditions (i) and (ii) of *Definition 1*.

In order to show that $\|x^r\| = r$, we examine the structure of the frontier ∂D_r. It is readily verified that

$$\partial D_r = V_r \cup U_\delta \tag{32}$$

where

$$V_r = \{x \in R^n \| r = \|x^+\| \leq \delta\} = \partial W_r \cap \bar{B}_\delta \tag{33}$$

and

$$U_\delta = W_r \cap S_\delta \tag{34}$$

Now we prove that $u_r \notin U_\delta$. Indeed, from (21) it follows that

$$\|u_r^-\| \leq \max\{\|u_r^+\|, \|f(u_r^+)\|\} \leq \max\{r, M_r\}.$$

Hence,

$$\|u_r\|^2 = \|u_r^+\|^2 + \|u_r^-\|^2 \leq r^2 + (\max\{r, M_r\})^2 = (\delta - 1)^2$$

which yields $\|u_r\| < \delta$.

Thus, $u_r \in V_r$ and consequently $\|x^r\| = \|u_r^+\| = r$. This means that $\{x^r\}_{r>0}$ is a regular exceptional family of elements, and the theorem is completely proved. ■

REMARK. We remark that *Theorem 6* can be derived by using only the *Hartman–Stampacchia Theorem [10]*. Indeed, suppose that $ECP(f, \mathbf{R}^n_+)$ has no solution. Then $VI(f, \mathbf{R}^n_+ \cap \{x : \|x\| \leq r\})$ has a solution, say x^r. We must have $\|x^r\| = r$ else, x^r becomes a solution of $ECP(f, \mathbf{R}^n_+)$. (If $\|x^r\| < r$, then for any $z \in \mathbf{R}^n_+$, $\|x^r + \varepsilon(z - x^r)\| < r$ for some $\varepsilon > 0$. The inequality $\langle f(x^r), x - x^r\rangle \geq 0$ for all $x \in \mathbf{R}^n_+ \cap \{x : \|x\| \leq r\}$ implies that $\langle f(x^r), z - x^r\rangle \geq 0$ for all $z \geq 0$, which says that x^r is a solution of $ECP(f, \mathbf{R}^n_+)$.). Now for the problem $\min\{\langle f(x^r), x\rangle : x \geq 0, \|x\| \leq r\}$ which takes its optimum value at x^r, the KKT conditions are precisely the conditions that appear in *Definitions 1 and 2* for a regular exceptional sequence. This proof is not applicable to *Theorems 7–11*. Because of this result, we use for all the theorems only the topological degree. The topological degree can be used also to extend our results to infinite dimensional spaces. The exceptional families of elements can also be used to study the implicit complementarity problem.

THEOREM 7. *Let $f, g : \mathbf{R}^n \to \mathbf{R}^n$ be continuous mappings. If the following assumptions are satisfied:*

(1) there exists $b \in \mathbf{R}^n$ such that $g(x) = 0$ if and only if $x = b$,

(2) g maps a neighborhood of the point b homeomorphically onto a neighborhood of the origin,

then there exists either a solution of the problem $ICP(f, g, \mathbf{R}^n_+)$ or an exceptional family of elements for the couple (f, g).

Proof. We consider the following equation with respect to the variable $(z, x) \in \mathbf{R}^n \times \mathbf{R}^n$:

$$F(z, x) = \begin{pmatrix} f(x) - z^- \\ g(x) - z^+ \end{pmatrix} = 0. \tag{35}$$

The problem $ICP(f, g, \mathbf{R}^n_+)$ is equivalent to the solvability of equation (35). Indeed, if (z, x) solves (35), then x is a solution of the problem $ICP(f, g, \mathbf{R}^n_+)$. Conversely, if x is a solution of the problem $ICP(f, g, \mathbf{R}^n_+)$ then (z, x) is a solution of (35) where

$$z_i = \begin{cases} g_i(x), \ \textit{if}\ g_i(x) > 0, \\ -f_i(x), \ \textit{if}\ g_i(x) = 0, \end{cases} \qquad i = 1, 2, 3, \ldots, n.$$

The mapping $F(z, x)$ is clearly continuous over $\mathbf{R}^{2n}$. Let S_r be a $(2n - 1)$-dimensional sphere:

$$S_r = \{(z, x) \in \mathbf{R}^{2n} | \|(z, x - b)\| = r\},$$

and B_r an open ball of radius r, i.e.

$$B_r = \{(z, x) \in \mathbf{R}^{2n} | \|(z, x - b)\| < r\},$$

Further, we construct a homotopy $H(z, x, t)$ of the mappings $F(z, x)$ and

$$G(z, x) = \begin{pmatrix} z \\ g(x) \end{pmatrix}$$

in the standard way:

$$\begin{aligned} H(z, x, t) &= tG(z, x) + (1 - t)F(z, x) \\ &= \begin{pmatrix} tz + (1 - t)f(x) - (1 - t)z^- \\ tg(x) + (1 - t)g(x) - (1 - t)z^+ \end{pmatrix} = \begin{pmatrix} tz^+ + (1 - t)f(x) - z^- \\ g(x) - (1 - t)z^+ \end{pmatrix}. \end{aligned}$$

Hence, we have

$$H(z, x, t) = \begin{pmatrix} tz^+ + (1 - t)f(x) - z^- \\ g(x) - (1 - t)z^+ \end{pmatrix}. \tag{36}$$

Two cases are possible:
(A). There exists an $r > 0$ such that

$$H(z, x, t) \neq 0 \text{ for all } (z, x) \in S_r \text{ and } t \in [0, 1].$$

The Poincaré–Bohl Theorem implies the equality

$$\deg(F, B_r, 0) = \deg(G, B_r, 0).$$

Since $|\deg(G, B_r, 0)| = 1$, we have that $\deg(F, B_r, 0) = \pm 1$.

As above, we conclude that the ball $\bar{B}_r$ contains at least one solution of the equation $F(z, x) = 0$ and the solvability of the problem $ICP(f, g, \mathbf{R}^n_+)$ is proved.
(B). For $r > 0$ there exist a point $(z_r, x^r) \in S_r$ and a scalar $t_r \in [0, 1)$ such that

$$H(z_r, x^r, t_r) = 0. \tag{37}$$

We remark that

$$\|(z_r, x^r - b)\|_{2n}^2 = \|z_r^+\|_n^2 + \|z_r^-\|_n^2 + \|x^r - b\|_n^2 = r^2. \tag{38}$$

If $t_r = 0$, then (z_r, x^r) solves the equation (35) and hence, x^r solves the problem $ICP(f, g, \mathbf{R}_+^n)$.

Otherwise, i.e. if $t_r > 0$, from (36) and (37) we obtain

$$t_r z_r^+ + (1 - t_r) f(x^r) = z_r^-, \tag{39}$$

$$z_r^+ = \frac{g(x^r)}{1 - t_r}. \tag{40}$$

Substituting expression (40) for z_r^+ into (39) yields

$$\frac{t_r}{1 - t_r} g(x^r) + (1 - t_r) f(x^r) = z_r^-$$

which implies for $i = 1, 2, \ldots, n$

$$f_i(x^r) = \begin{cases} \dfrac{t_r g(x^r)}{(1 - t_r)^2}, & \textit{if } (z_r)_i > 0 \\ \dfrac{(z_r^-)_i}{(1 - t_r)}, & \textit{if } (z_r)_i \leq 0. \end{cases} \tag{41}$$

Taking $\mu_r = \frac{t_r}{(1-t_r)^2}$ we note that (41) guarantees the family of points $\{x^r\}$ to be exceptional if $\|x^r\| \to +\infty$ when $r \to +\infty$. To prove the last relation, we suppose on the contrary that the family $\{x^r\}$ has a finite cluster point $\bar{x}$ and remark that the corresponding cluster point $\bar{t}$ cannot be equal to 1 (otherwise (38) and (39) contradict each other).

Then the continuity of mappings f and g together with (39) and (40) imply the boundedness of the family $\{z_r\}$, which again contradicts (38), as $r \to +\infty$. This completes the proof of *Theorem 7*. ∎

We apply now the concept of exceptional family of elements to the study of the complementarity problem with respect to an arbitrary convex cone in the Euclidean space.

Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a continuous function and $\mathbf{K} \subset \mathbf{R}^n$ be a closed convex cone. As indicated in the section on preliminaries, **K** is a pointed convex cone. In the next result we use the concept of exceptional family of elements defined by *Definition 3*. We have the following result.

THEOREM 8. *If $f : \mathbf{R}^n \to \mathbf{R}^n$ is a continuous function and $\mathbf{K} \subset \mathbf{R}^n$ is a closed pointed convex cone, then there exists either a solution for the problem $ECP(f, \mathbf{K})$ or an exceptional family of elements for f.*

Proof. Using *Proposition 2* and *Moreau's Theorem* we have that the solvability of the problem $ECP(f, \mathbf{K})$ is equivalent to the solvability of the equation

$$F(x) = f(x^+) - x^- = 0, \tag{42}$$

where $x^+ = P_{\mathbf{K}}(x)$ and $x^- = -P_{\mathbf{K}^0}(x)$.

Repeating now exactly the proof of *Theorem 5* we obtain that either there exists a solution for the problem $ECP(f, \mathbf{K})$, or for each $r > 0$ there exist a point $u_r \in S_r$ and a scalar $t_r \in (0, 1)$ such that the equality

$$t_r u_r^+ + (1 - t_r) f(u_r^+) = u_r^-$$

is true.

Dividing both sides of that equality by $(1 - t_r)$ and rearranging, one obtains the relation

$$f(u_r^+) = \frac{1}{1 - t_r}\, u_r^- - \frac{t_r}{1 - t_r} u_r^+ = u_r^- + \frac{t_r}{1 - t_r}(u_r^- - u_r^+), \tag{43}$$

which means that $f(u_r^+) \in \mathcal{O}(u_r^-; s_r)$.

The fact that $\|u_r^+\| \to +\infty$ as $r \to +\infty$ is established in the same way as in the proof of *Theorem 5*.

Thus $\{x^r\}_{r>0}$, where $x^r = u_r^+$ is an exceptional family of elements for f (with respect to $\mathbf{K}$) and this completes the proof. ■

By an argument similar to the proof of *Theorem 6* we obtain the following result.

THEOREM 9. *For any continuous mapping $f : \mathbf{R}^n \to \mathbf{R}^n$ and any closed point convex cone $\mathbf{K} \subset \mathbf{R}^n$, there exists either a solution for the problem $ECP(f, \mathbf{K})$, or a regular exceptional family of elements for f.*

In [6], [7], [12] interesting relations between the problems $GOCP(\{f_i\}_{i=1}^n, \mathbf{K})$ and $ICP(f, \mathbf{K})$ are established.

In this way we can apply the concept of *exceptional family of elements* to the study of the problem $GOCP(\{f_i\}_{i=1}^n, \mathbf{K})$.

Indeed, it is well known that if $(\mathbf{R}^n, \langle, \rangle, \mathbf{K})$ is a Hilbert lattice with respect to the ordering defined by $\mathbf{K}$ and if $x, y \in \mathbf{K}$, then $x \wedge y = 0$ if and only if $\langle x, y \rangle = 0$. We have the following result.

THEOREM 10. *Let $f_1, f_2, \ldots, f_m$ be continuous mappings from $\mathbf{R}^n$ into $\mathbf{R}^\mathbf{n}$. If the following assumptions are satisfied:*

1. *there exists $b \in \mathbf{R}^n$ such that $f_1(x) = 0$ if and only if $x = b$,*
2. *f_1 maps a neighborhood of the point b homeomorphically onto a neighborhood of the origin,*

then there exists either a solution of the problem $GOCP(\{f_i\}_{i=1}^m, \mathbf{R}_+^n)$ or an exceptional family of elements for the couple (h, f_1) where $h(x) = \wedge(f_2(x), f_3(x), \ldots, f_m(x))$ for each $x \in \mathbf{R}^n$.

Proof. Since $(\mathbf{R}^n, \langle,\rangle, \mathbf{R}^n_+)$ is a Hilbert lattice we have that the problem $GOCP$ $(\{f_i\}_{i=1}^m, \mathbf{R}^n_+)$ is equivalent to the problem $ICP(h, f_1, \mathbf{R}^n_+)$. The conclusion of the theorem follows from *Theorem 7,* since we can show that h is a continuous mapping. ∎

REMARK. Given $f_1, f_2, f_3, \ldots, f_m$ continuous mappings from $\mathbf{R}^n$ into $\mathbf{R}^n$, and considering the mapping $h(x) = \wedge(f_2(x), \ldots, f_m(x))$ it is useful to know under what conditions the couple (h, f_1) is without exceptional families of elements.

COROLLARY. If the mappings $f_1, f_2, f_3, \ldots, f_m$, from $\mathbf{R}^n$ and $\mathbf{R}^n$, are continuous, f_1 satisfies the assumptions 1) and 2) of Theorem 10 and the couple (h, f_1) is without exceptional families of elements, where $h = \wedge(f_2, \ldots, f_m)$, then the problem $GOPC(\{f_1\}_{i=1}^m, \mathbf{R}^n_+)$ has a solution.

Certainly, a natural question is to know if the results presented in this apper are true in an arbitrary infinite dimensional Hilbert space. In this sense we have the next result for cones with compact base. The cones with compact base are studied in [22].

THEOREM 11. *Let $(H, \langle,\rangle)$ be an arbitrary Hilbert space, $\mathbf{K} \subset H$ a closed pointed convex cone and $f : H \to H$ a continuous function. If $\mathbf{K}$ has a compact base and f is a completely continuous operator, then there exists either a solution for the problem $ECP(f, \mathbf{K})$ or an exceptional family of elements for f.*

Proof. Indeed, the solvability of problem $ECP(f, \mathbf{K})$ is equivalent to the solvability of equation

$$F(x) = x - P_{\mathbf{K}}(x) + f(P_{\mathbf{K}}(x)) = 0 \tag{44}$$

or

$$F(x) = x - [P_{\mathbf{K}}(x) - f(P_{\mathbf{K}}(x))] = 0. \tag{45}$$

Since $\mathbf{K}$ has a compact base, it is a locally compact cone, which implies that $\mathbf{P_K}$ is a completely continuous operator. We have that F is a Leray–Schauder map and hence, the topological degree is well defined [23], [16].

The proof follows the proof of *Theorem 8,* but using in this case the *Poincaré–Bohl's Theorem* as presented in the book [23]. (*Theorem 1* must be also used). ∎

OPEN PROBLEMS. The following open problems can be considered:

(1) Find a test or an algorithm which can be used to decide whether or not a function is without exceptional families of elements.
(2) Study the relations between the property of a function f to be without exceptional families of elements when f is an affine function and the classes of matrices studied by C.B. Garcia in [2].
(3) Generalize the results presented in this paper to the infinite dimensional case.

Conclusions

Considering our results, presented in this paper, we suggest that the concept of *exceptional family of elements* associated to a continuous function opens a new interesting research direction in the complementarity theory. The relation between this concept and the topological degree is a sufficient argument to conclude that the property to be *without exceptional family of elements* is a deep property of continuous functions. Probably, this property can be considered as a substantial generalization of the concept of *coercive function*.

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